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Formal deformations of Dirac structures

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Abstract

In this paper we set-up a general framework for a formal deformation theory of Dirac structures. We give a parameterization of formal deformations in terms of two-forms obeying a cubic equation. The notion of equivalence is discussed in detail. We show that the obstruction for the construction of deformations order by order lies in the third Lie algebroid cohomology of the Dirac structure. However, the classification of inequivalent first order deformations is not given by the second Lie algebroid cohomology but turns out to be more complicated.

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1. Introduction

A Dirac structure is a maximally isotropic subbundle of a Courant algebroid whose sections in addition are closed under the Courant bracket. A Courant algebroid is a vector bundle with a not necessarily positive definite fiber metric over a base manifold which is equipped with a bundle map into the tangent bundle (the anchor) and a bracket on its sections, the Courant bracket, subject to certain compatibility conditions. The fundamental example of a Courant algebroid is $E = TM \oplus T^*M$ with the natural pairing as fiber metric, the identity on the first component as anchor and the bracket

$$[(X, \alpha), (Y, \beta)]_{\mathcal{C}} = ([X, Y], \mathcal{L}_X \beta - i_Y \mathrm{d}\alpha).$$

Then both TM and T^*M are Dirac structures in this Courant algebroid.

Dirac structures were introduced by Courant [7] to generalize on the one hand symplectic and Poisson structures, and on the other to provide powerful tools to describe dynamics subject to constraints. Moreover, they can also be used

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to encode various 'oid'-structures, in particular Lie bialgebroids [24–26]. For the general notions of Dirac structures we refer to [7,24].

As Dirac structures combine symplectic and Poisson structures and can be used in constraint dynamics, it is natural to ask what a physically reasonable quantization of a Dirac structure should be. In particular, this could shine some new light on the quantization of constraint dynamics and phase space reduction. In deformation quantization [1], see e.g. [10,16,35] for recent reviews, one knows that the equivalence classes of formal quantizations of Poisson structures are in one-to-one correspondence with equivalence classes of formal deformations of the given Poisson structure into formal Poisson structures modulo formal diffeomorphisms. This is one of the main corollaries of Kontsevich's formality theorem [18].

Motivated by this result, we investigate the deformation theory of Dirac structures in order to determine their *classical deformations* into formal Dirac structures up to formal diffeomorphisms. We hope that this will give eventually some hints on how to formulate a definition of deformation quantization of Dirac structures such that a type of formality might hold true also in this context. The first steps in this direction have been taken in [32] by Ševera who proposed a deformation quantization of formal deformations of regular Dirac structures. Note however, that the classical deformations are also of interest if one wants to describe stability/rigidity of Dirac structures, not necessarily aiming at quantization. Thus our first aim in this paper is to set-up a reasonable definition of a formal Dirac structure and investigate basic properties of the corresponding classical deformation theory.

As a Dirac structure L gives in particular the structure of a Lie algebroid it is natural to compare the formal deformation theory of Dirac structures with the formal deformation theory of L as a Lie algebroid in the sense of [9]: It turns out that any deformation of a Dirac structure induces a Lie algebroid deformation, but not necessarily vice-versa. Moreover, in [9] it was shown that the Lie algebroid structure of TM is rigid with respect to formal deformations while it is easy to see that this is not the case for Dirac structure deformations, here any non-trivial pre-symplectic form provides a non-trivial deformation.

The main results of this paper are, on the one hand, that the obstruction space for formal order-by-order deformations of a Dirac structure L is given by the third Lie algebroid cohomology of L; on the other, and this is the more surprising result, that the reasonable notion of equivalence up to formal diffeomorphisms does *not* yield a classification of inequivalent first order deformations in the second Lie algebroid cohomology, as one might first think: the actual classification is more involved and seems to be beyond a simple cohomological formulation. This depends of course on our definition of equivalence which we based on formal diffeomorphisms. Most of our results emerged from the Diplomarbeit [17].

As a main technique it turned out that a description of formal Dirac structures in terms of graphs of formal two-forms requires some reasonable calculus. We found the derived bracket formalism [19,20], already introduced by Roytenberg in a super-geometric way [29], most useful. However, we realized the derived bracket formalism not in terms of super-geometry but used more conventional geometric objects: the main ingredient is the Rothstein–Poisson bracket [28]. We believe that this approach has its own interest, in particular when it comes to quantization as we can rely on Bordemann's results for the deformation quantization of the Rothstein–Poisson bracket [2,3]. Nevertheless, our approach is completely equivalent to the one of Roytenberg.

The paper is organized as follows: In Section 2 we recall some basic definitions and results on Courant algebroids, their automorphisms and their Dirac structures. Section 3 introduces the derived bracket point of view in order to handle the quite complicated algebraic identities of the Courant bracket in a more efficient way. We recall the Rothstein–Poisson bracket and use it to formulate Dirac structures in this context. In Section 4 we first formulate a smooth deformation of a Dirac structure and discuss the problem of equivalence up to diffeomorphisms for the case of a general Courant algebroid. Taking this as motivation we pass to formal deformations by Taylor expansion in the deformation parameter as usual. The fundamental equation, a sort of Maurer–Cartan equation which controls the deformation, has already been discussed in some different contexts in [31, Eq. (4.3)]. We show that the order-by-order construction of a formal deformation yields obstructions in the third Lie algebroid cohomology of the undeformed Dirac structure. Finally, we discuss the notion of equivalence up to formal diffeomorphisms in detail and point out that the second Lie algebroid cohomology is not necessarily the space of inequivalent first order deformations. Finally, Appendix A gives an overview on the Rothstein–Poisson bracket and recalls some of its basic properties.

Conventions: Throughout the paper we use Einstein's summation convention, i.e. summation over repeated coordinate indices is automatic.

2. General remarks on Dirac structures in Courant algebroids

In this section we recall some basic notions of Courant algebroids and Dirac structures in order to set up our notation. Most of the material is standard, see e.g. [7,24,29].

2.1. Courant algebroids

Definition 2.1. A Courant algebroid is a vector bundle $E \longrightarrow M$ together with a nondegenerate symmetric bilinear form h, a bracket $[\cdot,\cdot]_{\mathcal{C}} : \Gamma^\infty(E) \times \Gamma^\infty(E) \longrightarrow \Gamma^\infty(E)$ on the sections of the bundle and a vector bundle homomorphism $\rho : E \longrightarrow TM$, called the anchor, such that for all $e_1, e_2, e_3 \in \Gamma^\infty(E)$ and $f \in C^\infty(M)$ the following conditions hold:

(i) Jacobi identity, i.e. $[e_1, [e_2, e_3]_{\mathcal{C}}]_{\mathcal{C}} = [[e_1, e_2]_{\mathcal{C}}, e_3]_{\mathcal{C}} + [e_2, [e_1, e_3]_{\mathcal{C}}]_{\mathcal{C}}$,

(ii) $[e_1, e_2]_{\mathcal{C}} + [e_2, e_1]_{\mathcal{C}} = \mathcal{D}\, h(e_1, e_2)$, where $\mathcal{D} : C^\infty(M) \longrightarrow \Gamma^\infty(E)$ is defined by
$$h(\mathcal{D} f, e) = \rho(e) f,$$

(iii) $\rho(e_1)h(e_2, e_3) = h([e_1, e_2]_{\mathcal{C}}, e_3) + h(e_2, [e_1, e_3]_{\mathcal{C}})$.

An easy computation shows that the Courant bracket $[\cdot,\cdot]_{\mathcal{C}}$ satisfies the Leibniz rule

$$[e_1, f e_2]_{\mathcal{C}} = f[e_1, e_2]_{\mathcal{C}} + (\rho(e_1) f) e_2 \tag{2.1}$$

and the anchor turns out to satisfy

$$\rho([e_1, e_2]_{\mathcal{C}}) = [\rho(e_1), \rho(e_2)] \tag{2.2}$$

for all $e_1, e_2, e_3 \in \Gamma^\infty(E)$ and $f \in C^\infty(M)$, see e.g. [21,22,34].

Remark 2.2. Equivalent to this definition is the one given in [6]. One can also consider the object obtained by skew-symmetrization of the Courant bracket, which is sometimes referred to as a Courant algebroid. Both definitions are equivalent, see [29] for a detailed discussion.

The above definition for a Courant algebroid is the generalization of an object studied by Courant in [7], which we will refer as the standard Courant algebroid:

Example 2.3 (*Standard Courant Algebroid [7]*). Consider for a manifold M the vector bundle $E = TM \oplus T^*M$. The canonical symmetric bilinear form on E given by

$$\langle (X, \alpha), (Y, \beta) \rangle = \alpha(Y) + \beta(X), \tag{2.3}$$

where $X, Y \in \mathfrak{X}(M)$ and $\alpha, \beta \in \Omega^1(M)$, together with the bracket

$$[(X, \alpha), (Y, \beta)]_{\mathcal{C}} = ([X, Y], \mathcal{L}_X \beta - i_Y \mathrm{d}\alpha) \tag{2.4}$$

and the anchor ρ defined by $\rho(X, \alpha) = X$ endows E with the structure of a Courant algebroid.

Remark 2.4. According to our definition of a Courant algebroid we use here also the non skew-symmetric version where originally in [7] the skew-symmetric bracket was used.

Other examples for Courant algebroids are given by the double of Lie bialgebroids [24], or more generally by the doubles of Lie quasi-bialgebroids or proto bialgebroids, see [21]. We will come back to these examples later.

2.2. Automorphisms of Courant algebroids

Crucial for our investigations of deformations of Dirac structures will be an appropriate notion of isomorphism. To this end we need the automorphisms of the Courant algebroid. If $E \longrightarrow M$ is a Courant algebroid, then a vector bundle automorphism $\Phi : E \longrightarrow E$ over a diffeomorphism $\phi : M \longrightarrow M$ is called an automorphism of the Courant algebroid, if the following two conditions are fulfilled: First, Φ is an isometry of the bilinear form h, i.e. for all $e_1, e_2 \in \Gamma^\infty(E)$

$$h(\Phi^* e_1, \Phi^* e_2) = \phi^*(h(e_1, e_2)). \tag{2.5}$$

Second, Φ is natural with respect to the Courant bracket, i.e. for all $e_1, e_2 \in \Gamma^\infty(E)$

$$[\Phi^* e_1, \Phi^* e_2]_{\mathcal{C}} = \Phi^*[e_1, e_2]_{\mathcal{C}}. \tag{2.6}$$

The following lemma shows that the compatibility with the anchor is already fixed by these two conditions:

Lemma 2.5. *If $\Phi : E \longrightarrow E$ is a Courant algebroid automorphism then the anchor ρ satisfies*

$$\rho \circ \Phi = T\phi \circ \rho. \tag{2.7}$$

Proof. This is used implicitly in [15, Prop. 3.24]: Using (2.1) and then (2.6) gives $[\Phi^* e_1, \Phi^*(f e_2)]_{\mathcal{C}} = \Phi^*(f[e_1, e_2]_{\mathcal{C}}) + \rho(\Phi^* e_1)(\phi^* f)\Phi^* e_2$. The other way round gives $[\Phi^* e_1, \Phi^*(f e_2)]_{\mathcal{C}} = \Phi^*(f[e_1, e_2]) + \phi^*(\rho(e_1) f)\Phi^* e_2$ whence we obtain

$$\rho(\Phi^* e_1)(\phi^* f) = \phi^*(\rho(e_1) f) = \phi^*(\rho(e_1))(\phi^* f)$$

for all $e_1, e_2 \in \Gamma^\infty(E)$ and $f \in C^\infty(M)$, which implies (2.7). ∎

In the case of the standard Courant algebroid one can determine the group of automorphisms completely. We recall the following definition [33]:

Definition 2.6 (*Gauge Transformations*). Let $E = TM \oplus T^*M$ be the standard Courant algebroid and $B \in \Omega^2(M)$ a two-form. A gauge transformation is a map $\tau_B : TM \oplus T^*M \longrightarrow TM \oplus T^*M$ given by $\tau_B(X, \alpha) = (X, \alpha + i_X B)$.

Lemma 2.7 (*Ševera, Weinstein [33]*). *A gauge transformation τ_B is an automorphism of the standard Courant algebroid structure on $TM \oplus T^*M$ if and only if B is closed.*

Let ϕ be a diffeomorphism of M. Then we denote the canonical lift of ϕ to $TM \oplus T^*M$ by $\mathcal{F}\phi = (T\phi, T_*\phi)$, where $T_*\phi : T^*M \longrightarrow T^*M$ is given by $T_*\phi(\alpha_p) = (T\phi^{-1})^*\alpha_p = \alpha_p \circ T_{\phi(p)}\phi^{-1}$ for $\alpha_p \in T_p^*M$. We further write $\mathcal{B}\phi$ for the inverse of $\mathcal{F}\phi$. With this notation, the following proposition describes all automorphisms of the standard Courant algebroid, see [15, Prop. 3.24]:

Proposition 2.8. *Let $E = TM \oplus T^*M$ be the standard Courant algebroid. Then every automorphism Φ of E is of the form*

$$\Phi = \tau_B \circ \mathcal{F}\phi, \tag{2.8}$$

with a unique closed 2-form $B \in \Omega^2(M)$ and an unique diffeomorphism $\phi : M \longrightarrow M$. The automorphism group of $TM \oplus T^*M$ is given by the semi-direct product $\mathcal{Z}^2(M) \rtimes \mathcal{D}iff(M)$ with $\mathcal{Z}^2(M) = \ker(\mathrm{d}_{|\Omega^2(M)})$, where the group multiplication is

$$(B, \phi)(C, \psi) = (B + (\phi^{-1})^* C, \phi \circ \psi). \tag{2.9}$$

2.3. Dirac structures

The definition of Dirac structures on manifolds is due to Courant [7] and was later generalized to Courant algebroids in [24]:

Definition 2.9. Let E be a Courant algebroid. A subbundle $L \subset E$ is called a Dirac structure if L is maximally isotropic with respect to the given bilinear form and if $\Gamma^\infty(L)$ is closed under the Courant bracket, i.e. $[\Gamma^\infty(L), \Gamma^\infty(L)] \subseteq \Gamma^\infty(L)$.

In the following, whenever we speak about Courant algebroids with Dirac structures, we will restrict ourself to Courant algebroids with *even fiber dimension* and a bilinear form of *signature zero*. The reason for this is that maximal isotropic subbundles in such Courant algebroids have half the fiber dimension of the algebroid, a point that will become important later on.

In particular, the standard Courant algebroid $TM \oplus T^*M$ is of this type. In this case, one has the following two standard examples of Dirac structures:

Example 2.10. Let $E = TM \oplus T^*M$ be the standard Courant algebroid over M.

(i) Given a two-form $\omega \in \Omega^2(M)$, we consider ω as a map $\omega : TM \longrightarrow T^*M$ by defining

$$\omega(X) = i_X\omega = \omega(X, \cdot). \tag{2.10}$$

Thanks to skew-symmetry of ω the $\dim M$-dimensional subbundle $L := \operatorname{graph}(\omega) \subset TM \oplus T^*M$ is isotropic. Moreover L is closed under the Courant bracket, i.e. is a Dirac structure, if and only if ω is closed. Thus presymplectic two-forms can be viewed as particular cases of Dirac structures.

(ii) Let $\pi \in \Gamma^\infty(\bigwedge^2 TM)$ be a bivector. We consider π as a map $\pi : T^*M \longrightarrow TM$ by defining

$$\pi(\alpha) = \pi(\alpha, \cdot). \tag{2.11}$$

Again due to skew-symmetry $L := \operatorname{graph}(\pi) \subset TM \oplus T^*M$ is a maximal isotropic subbundle. One further finds that L is a Dirac structure if and only if π is a Poisson tensor, i.e. $[\pi, \pi] = 0$.

3. Derived brackets for Courant algebroids and Dirac structures

In this section we shall realize the Courant bracket as a derived bracket in the sense of [19,20] as this has been done before by Roytenberg [29,30] in a slightly different context.

3.1. The Rothstein–Poisson bracket

For the study of Poisson manifolds the Schouten–Nijenhuis bracket has turned out to be a very useful tool since one can write the Poisson bracket as a derived bracket $\{f, g\} = -[[f, \pi], g]$ for a unique bivector $\pi \in \Gamma^\infty(\bigwedge^2 TM)$. It then follows immediately that the Jacobi identity for the Poisson bracket is equivalent to the equation $[\pi, \pi] = 0$, see e.g. [20] for an overview on derived brackets. In the case of a Courant algebroid E a similar approach is possible. However, one first has to find an appropriate space which has the sections $\Gamma^\infty(E)$ as a subset as well as a bracket on it, in order to write the Courant bracket as a derived bracket. One possibility favored by Roytenberg [29,30] is given by the space of functions on a suitable symplectic supermanifold.

We shall use a slightly different presentation avoiding the explicit notion of supermanifolds: in our approach we take advantage of more conventional differential geometry by using the Rothstein–Poisson bracket [28] on the sections of the Grassmann algebra of the pullback bundle $\tau^\# E \longrightarrow T^*M$, see Appendix A for precise definitions. The Rothstein–Poisson bracket satisfies a graded Leibniz rule with respect to the $\wedge$-product, is graded antisymmetric and fulfilles a graded Jacobi identity where all signs come from the Grassmann parity. Though the structure is essentially the same as in [29,30], which can made even more transparent in the super-Darboux coordinates from Appendix A.3, the explicit use of ordinary differential geometry might come in useful when considering a quantized version of Dirac structures as we can rely on e.g. Bordemann's construction [2,3] for deformation quantization of the Rothstein–Poisson bracket. Furthermore, the usage of the Rothstein–Poisson bracket allows us to perform intrinsically *global* computations.

Let $E \longrightarrow M$ be a vector bundle together with a fiber metric h, i.e. a nondegenerate bilinear form, and let ∇ be a metric connection on E. We denote by $\tau : T^*M \longrightarrow M$ the cotangent bundle. Then on the supercommutative algebra $\Gamma^\infty\left(\bigwedge^\bullet \tau^\# E\right)$ of sections of the pulled back bundle $\tau^\# E \longrightarrow T^*M$ we have the Rothstein–Poisson bracket as described in Appendix A.2, defined by use of the pull back of the fiber metric h and the connection ∇. We can regard $\Gamma^\infty\left(\bigwedge^\bullet E\right)$ as a subalgebra of $\Gamma^\infty\left(\bigwedge^\bullet \tau^\# E\right)$ via the pull-back of sections.

Since T^*M is a vector bundle itself and since we consider a pulled back bundle over T^*M, it makes sense to speak of sections $e \in \Gamma^\infty\left(\bigwedge^\bullet \tau^\# E\right)$ which are polynomial in the fiber directions of T^*M of degree $k \in \mathbb{N}$. Note that the grading with respect to the fiber variables (the momenta) is not a good grading for the Rothstein–Poisson bracket, neither is the Grassmann degree. However, the Rothstein–Poisson bracket is graded with respect to twice the polynomial degree in the momenta plus the Grassmann degree. We denote homogeneous sections of this *total degree* $k \in \mathbb{N}$ by $\mathcal{P}^k \subseteq \Gamma^\infty\left(\bigwedge^\bullet \tau^\# E\right)$. Then their direct sum $\mathcal{P}^\bullet$ is a subalgebra of $\Gamma^\infty\left(\bigwedge^\bullet \tau^\# E\right)$, both with respect to the $\wedge$-product and the Rothstein bracket. With respect to this grading, the Rothstein–Poisson bracket has degree -2, i.e.

$$\{\mathcal{P}^k, \mathcal{P}^\ell\}_{\mathcal{R}} \subseteq \mathcal{P}^{k+\ell-2}. \tag{3.1}$$

In particular, $\tau^* C^\infty(M) = \mathcal{P}^0$ and $\tau^\# \Gamma^\infty(E) = \mathcal{P}^1$, see Appendix A.4.

3.2. Courant algebroids via Rothstein bracket

With the help of the Rothstein–Poisson bracket on $\Gamma^\infty\left(\bigwedge^\bullet \tau^\# E\right)$ we can define a derived bracket [19] on $\Gamma^\infty(E)$. Consider for $\Theta \in \Gamma^\infty\left(\bigwedge^\bullet \tau^\# E\right)$ the bilinear map on $\Gamma^\infty\left(\bigwedge^\bullet \tau^\# E\right)$ given by

$$(\xi, \zeta) \longmapsto \{\{\xi, \Theta\}_{\mathcal{R}}, \zeta\}_{\mathcal{R}}. \tag{3.2}$$

In order to get a derived bracket on $\Gamma^\infty(E)$, the subspace $\mathcal{P}^1 = \Gamma^\infty(E)$ has to be closed under the above map. As one can see in the local formula (A.12) for the Rothstein–Poisson bracket this is only the case for a homogeneous $\Theta \in \mathcal{P}^3$ of total degree 3. In fact, the lower degrees do not contribute and higher ones will not produce pullbacks of sections from $\Gamma^\infty(E)$. Such a section $\Theta \in \mathcal{P}^3$ has two types of contributions: one is a section of $\Gamma^\infty(\tau^\# E)$ which is *linear* in the momenta variables of T^*M, the other is a pull-back section of $\Gamma^\infty(\bigwedge^3 E)$.

Lemma 3.1. *Let $\Theta \in \mathcal{P}^3$. Then the following objects are well-defined:*

(i) *A $\mathbb{R}$-bilinear derived bracket $[\cdot,\cdot]_\Theta : \Gamma^\infty(E) \times \Gamma^\infty(E) \longrightarrow \Gamma^\infty(E)$ defined for $e_1, e_2 \in \Gamma^\infty(E)$ by*

$$[e_1, e_2]_\Theta = \{\{e_1, \Theta\}_{\mathcal{R}}, e_2\}_{\mathcal{R}}. \tag{3.3}$$

(ii) *A derived anchor, i.e. a bundle map $\rho_\Theta : E \longrightarrow TM$ defined for $e \in \Gamma^\infty(E)$ and $f \in C^\infty(M)$ by*

$$\rho_\Theta(e) f = \{\{e, \Theta\}_{\mathcal{R}}, f\}_{\mathcal{R}}. \tag{3.4}$$

(iii) *A map $\mathcal{D}_\Theta : C^\infty(M) \longrightarrow \Gamma^\infty(E)$ defined for $f \in C^\infty(M)$ by*

$$\mathcal{D}_\Theta f = \{\Theta, f\}_{\mathcal{R}}. \tag{3.5}$$

The bundle E together with the bilinear form h and the above defined bracket and anchor satisfy the conditions (ii) *and* (iii) *from Definition* 2.1 *of a Courant algebroid.*

Proof. The well-definedness follows from the grading properties. Then the verification of the conditions (ii) and (iii) is a straightforward computation using the graded Jacobi identity of $\{\cdot,\cdot\}_{\mathcal{R}}$. ■

Note that the definition of $\mathcal{D}_\Theta$ is consistent with Definition 2.1. The following lemma is the analogue of [30] for the Rothstein–Poisson bracket and follows the general ideas of derived brackets [20].

Lemma 3.2. *Let $\Theta \in \mathcal{P}^3 \subset \Gamma^\infty\left(\bigwedge^\bullet(\tau^\# E)\right)$ be homogeneous of degree* 3. *Then E together with the bilinear form h, the bracket $[\cdot,\cdot]_\Theta$ and the anchor ρ_Θ is a Courant algebroid if and only if*

$$\{\Theta, \Theta\}_{\mathcal{R}} = 0. \tag{3.6}$$

Proof. For the 'if' part we only have to check the Jacobi identity for $[\cdot,\cdot]$, which is a simple computation in the framework of derived brackets [19]. We only have to use the graded Jacobi identity of $\{\cdot,\cdot\}_{\mathcal{R}}$. For the 'only if' part we assume the Jacobi identity for $[\cdot,\cdot]_\Theta$. Then

$$\{\{\{\{\Theta, \Theta\}_{\mathcal{R}}, e_1\}_{\mathcal{R}}, e_2\}_{\mathcal{R}}, e_3\}_{\mathcal{R}} = 0 \tag{$*$}$$

for all $e_1, e_2, e_3 \in \Gamma^\infty(E)$. Let $f \in C^\infty(M)$ be a function. Then by the graded Leibniz rule for $\{\cdot,\cdot\}_{\mathcal{R}}$ we have

$$\begin{aligned} 0 &= \{\{\{\{\Theta, \Theta\}_{\mathcal{R}}, e_1\}_{\mathcal{R}}, e_2\}_{\mathcal{R}}, f e_3\}_{\mathcal{R}} \\ &= f\{\{\{\{\Theta, \Theta\}_{\mathcal{R}}, e_1\}_{\mathcal{R}}, e_2\}_{\mathcal{R}}, e_3\}_{\mathcal{R}} + \{\{\{\Theta, \Theta\}_{\mathcal{R}}, e_1\}_{\mathcal{R}}, e_2\}_{\mathcal{R}}, f\}_{\mathcal{R}} e_3 \end{aligned}$$

from which we obtain

$$\{\{\{\{\Theta, \Theta\}_{\mathcal{R}}, e_1\}_{\mathcal{R}}, e_2\}_{\mathcal{R}}, f\}_{\mathcal{R}} = 0. \tag{$**$}$$

By another application of the graded Jacobi identity we also find

$$\{\{\{\Theta, \Theta\}_{\mathcal{R}}, \{e_1, e_2\}_{\mathcal{R}}\}_{\mathcal{R}}, f\}_{\mathcal{R}} = 0$$

for all $e_1, e_2 \in \Gamma^\infty(E)$ and $f \in C^\infty(M)$. Since locally every function $g \in C^\infty(M)$ can be written as $g = \{e_1, e_2\}_{\mathcal{R}}$ with appropriate $e_1, e_2 \in \Gamma^\infty(E)$ we conclude

$$\{\{\{\Theta, \Theta\}_{\mathcal{R}}, f\}_{\mathcal{R}}, g\}_{\mathcal{R}} = 0 \tag{$***$}$$

for all $f, g \in C^\infty(M)$. From the explicit formulas for $\{\cdot, \cdot\}_{\mathcal{R}}$ we see that the properties $(*)$, $(**)$ and $(***)$ together imply that the homogeneous element $\{\Theta, \Theta\}_{\mathcal{R}}$ of degree 4 has to vanish. ■

In a next step we want to construct such an element Θ for a given Courant algebroid. We begin with the following easy lemma:

Lemma 3.3. *Let $(E, [\cdot, \cdot]_{\mathcal{C}}, \rho, h)$ be a Courant algebroid with a metric connection ∇. Then the map $T : \Gamma^\infty(E) \times \Gamma^\infty(E) \times \Gamma^\infty(E) \longrightarrow \mathbb{R}$ defined by*

$$T(e_1, e_2, e_3) = h(\nabla_{\rho(e_1)} e_2 - \nabla_{\rho(e_2)} e_1 - [e_1, e_2]_{\mathcal{C}}, e_3) + h(\nabla_{\rho(e_3)} e_1, e_2) \tag{3.7}$$

is a skew-symmetric 3-tensor $T \in \Gamma^\infty(\bigwedge^3 E^*)$.

Proof. The proof is a direct computation using the definition of a Courant algebroid and the fact that the connection is metric. ■

In some sense, T is the Courant algebroid version of the torsion of ∇. Let $u_1, \ldots, u_K$ be a local basis of sections of E with dual basis $u^1, \ldots, u^K$, defined on the domain of a local chart $x^1, \ldots, x^n$ of M. Then locally T is given by

$$T = \frac{1}{6} T_{ABC} u^A \wedge u^B \wedge u^C \tag{3.8}$$

where $T_{ABC} = T(u_A, u_B, u_C)$. Using the musical isomorphism $\sharp$ induced by the fiber metric h we obtain from $T \in \Gamma^\infty(\bigwedge^3 E^*)$ the tensor field $T^\sharp \in \Gamma^\infty(\bigwedge^3 E)$, locally given by

$$T^\sharp = \frac{1}{6} h^{AE} h^{BF} h^{CG} T_{ABC} u_E \wedge u_F \wedge u_G. \tag{3.9}$$

Using the structure functions $C_{AB}^C = \langle [u_A, u_B]_{\mathcal{C}}, u^C \rangle$ of the Courant bracket, the components $\rho^i = \mathrm{d}x^i \circ \rho$ of the anchor, and the Christoffel symbols Γ_{iB}^A of ∇ we obtain by a straightforward computation

$$T^\sharp = \frac{1}{2} h^{AD} h^{BE} \rho^i(u_A) \Gamma_{iB}^C u_C \wedge u_D \wedge u_E - \frac{1}{6} h^{AD} h^{BE} C_{AB}^C u_C \wedge u_D \wedge u_E. \tag{3.10}$$

The second tensor field we shall need is obtained as follows. Since the anchor can be viewed as $\rho \in \Gamma^\infty(E^* \otimes TM)$ we can use h to obtain a tensor field $\rho^\sharp \in \Gamma^\infty(E \otimes TM)$. In a second step we can view the tangent vector field part of $\rho^\sharp$ as a linear function on T^*M whence we end up with a section $\mathcal{J}(\rho^\sharp) \in \Gamma^\infty(\tau^\# E)$, polynomial in the momenta of degree 1. Here, $\mathcal{J} : \Gamma^\infty(\mathrm{S}^\bullet TM) \longrightarrow \mathcal{P}^\bullet(T^*M)$ denotes the canonical algebra isomorphism. Locally, $\mathcal{J}(\rho^\sharp)$ is given by

$$\mathcal{J}(\rho^\sharp) = h^{AC} p_i \rho^i(u_A) u_C, \tag{3.11}$$

where $p_1, \ldots, p_n$ are the canonically conjugate momenta on T^*M to the local coordinates $q^1 = \tau^* x^1, \ldots, q^n = \tau^* x^n$ induced by the local coordinates $x^1, \ldots, x^n$ on M.

Putting both together we obtain from the choice of a metric connection ∇ the homogeneous element

$$\Theta = -\mathcal{J}(\rho^\sharp) + T^\sharp \in \mathcal{P}^3 \subseteq \Gamma^\infty\left(\tau^\# \bigwedge^\bullet E\right) \tag{3.12}$$

of total degree 3. For later use we shall give yet another local expression for Θ, namely using the super-Darboux coordinates from Proposition A.4. By rearranging the local expressions for $\mathcal{J}(\rho^\sharp)$ and $T^\sharp$ we obtain

$$\Theta = -h^{AC} r_i \rho^i(u_A) u_C - \frac{1}{6} h^{AD} h^{BE} C^C_{AB} u_C \wedge u_D \wedge u_E. \tag{3.13}$$

The advantage will be the easy commutation relations between the r_i and the other local variables. It is also the direct analogue to the supergeometric formulation of Roytenberg, see [30, Eq. (4.7)]. Note however, that this splitting of Θ is *not* coordinate independent, i.e. the two parts are *not* tensor fields, contrary to the splitting (3.12).

Lemma 3.4. *Let $E \longrightarrow M$ be a Courant algebroid and chose a metric connection ∇. Define the element $\Theta \in \mathcal{P}^3$ by (3.12). Then the Courant bracket and the anchor of E coincide with the derived bracket and the derived anchor induced by the element $\Theta \in \mathcal{P}^3$. In particular, $\{\Theta, \Theta\}_{\mathcal{R}} = 0$.*

Proof. Using the super-Darboux coordinates this is a simple verification. The second statement follows directly from Lemma 3.2. ■

Now we can finally make contact to the supermanifold formulation of Roytenberg. Analogously to [30, Thm. 4.5] we obtain:

Theorem 3.5. *Let $E \longrightarrow M$ be a vector bundle with fiber metric h and metric connection ∇. Then the set of Courant algebroid structures on E is in one-to-one correspondence with the set of $\Theta \in \mathcal{P}^3$ such that $\{\Theta, \Theta\}_{\mathcal{R}} = 0$.*

3.3. The case $E = L \oplus L^*$

Consider the case $E = L \oplus L^*$ for a vector bundle L endowed with the natural pairing as fiber metric of signature zero. In the following we shall use a connection on L and the corresponding induced metric connection on $L \oplus L^*$. From this choice we obtain the Rothstein–Poisson bracket on $\Gamma^\infty\left(\bigwedge^\bullet \tau^\#(L \oplus L^*)\right)$, see also Appendix A.5. The splitting $E = L \oplus L^*$ induces a bigrading instead of our previous total degree: Indeed, we set $\deg_L$ to be the polynomial degree in the momenta plus the L-degree and $\deg_{L^*}$ is the polynomial degree in the momenta plus the L^*-degree. Then $\mathcal{P}^{(r,s)}$ denotes those elements in $\mathcal{P}^{r+s}$ of $\deg_L$-degree r and $\deg_{L^*}$-degree s. Using this direct sum decomposition one obtains the following, analogously to [31]:

Lemma 3.6. *Let $\Theta = \psi + \mu + \gamma + \phi \in \mathcal{P}$ be an element of total degree 3 with $\psi \in \mathcal{P}^{(0,3)}$, $\mu \in \mathcal{P}^{(1,2)}$, $\gamma \in \mathcal{P}^{(2,1)}$ and $\phi \in \mathcal{P}^{(3,0)}$. Then $\{\Theta, \Theta\}_{\mathcal{R}} = 0$ is equivalent to*

$$\{\mu, \psi\}_{\mathcal{R}} = 0 \tag{3.14}$$

$$\frac{1}{2}\{\mu, \mu\}_{\mathcal{R}} + \{\gamma, \psi\}_{\mathcal{R}} = 0 \tag{3.15}$$

$$\{\phi, \psi\}_{\mathcal{R}} + \{\mu, \gamma\}_{\mathcal{R}} = 0 \tag{3.16}$$

$$\frac{1}{2}\{\gamma, \gamma\}_{\mathcal{R}} + \{\mu, \phi\}_{\mathcal{R}} = 0 \tag{3.17}$$

$$\{\gamma, \phi\}_{\mathcal{R}} = 0. \tag{3.18}$$

A quasi-Lie algebroid is a vector bundle $A \longrightarrow M$ together with a $\mathbb{R}$-bilinear, skew symmetric bracket $[\,\cdot, \cdot\,]_A$ on $\Gamma^\infty(A)$ and a vector bundle homomorphism $\rho_A : A \longrightarrow TM$ such that for all $a_1, a_2 \in \Gamma^\infty(A)$ and $f \in C^\infty(M)$ the Leibniz rule

$$[a_1, f a_2]_A = f[a_1, a_2]_A + \rho_A(a_1) f\, a_2 \tag{3.19}$$

is satisfied. If in addition the Jacobi identity for the bracket $[\,\cdot, \cdot\,]_A$ is fulfilled, then A is a Lie algebroid. In this case the anchor ρ_A is a homomorphism of Lie algebras,

$$\rho_A([a_1, a_2]_A) = [\rho_A(a_1), \rho_A(a_2)]. \tag{3.20}$$

The construction of the Schouten–Nijenhuis bracket can be generalized to the quasi-Lie algebroid case by imposing the graded Leibniz rule. The graded skew-symmetry and Leibniz rule are still satisfied, the graded Jacobi identity holds if and only if A is a Lie algebroid. The Lie algebroid differential also generalizes to the quasi-Lie algebroid case, and we get a differential with square zero if and only if A is a Lie algebroid. Conversely the quasi-Lie algebroid structure can be retrieved from the Schouten–Nijenhuis bracket or from the differential, see e.g. [27]. Identifying our derived brackets in this situation gives the following explicit formulas:

Lemma 3.7. *Let $\Theta = \psi + \mu + \gamma + \phi \in \mathcal{P} \subset \Gamma^\infty\left(\bigwedge^\bullet \tau^\#(L \oplus L^*)\right)$ be as above, and let $[\,\cdot\,,\cdot\,]_\Theta$ and ρ_Θ be the derived bracket and anchor.*

(i) *The restriction of $[\,\cdot\,,\cdot\,]_\Theta$ to L with subsequent projection to L is given by the derived bracket with respect to μ, i.e. for all $s_1, s_2 \in \Gamma^\infty(L)$ we have*

$$\mathrm{pr}_L([s_1, s_2]_\Theta) = [s_1, s_2]_\mu = \{\{s_1, \mu\}_{\mathcal{R}}, s_2\}_{\mathcal{R}}. \tag{3.21}$$

Further, the restriction of the anchor to L is given by

$$\rho_\Theta(s) f = \rho_\mu(s) f = \{\{s, \mu\}_{\mathcal{R}}, f\}_{\mathcal{R}} \tag{3.22}$$

for $s \in \Gamma^\infty(L)$ and $f \in C^\infty(M)$.

(ii) *The bracket $[\,\cdot\,,\cdot\,]_\mu$ together with the anchor ρ_μ make L a quasi Lie-algebroid. The associated Schouten–Nijenhuis bracket is given by*

$$[P, Q]_\mu = \{\{P, \mu\}_{\mathcal{R}}, Q\}_{\mathcal{R}} \tag{3.23}$$

for $P, Q \in \Gamma^\infty(\bigwedge^\bullet L)$, and the Lie algebroid differential by

$$\mathrm{d}_L \eta = \{\mu, \eta\}_{\mathcal{R}}, \tag{3.24}$$

where $\eta \in \Gamma^\infty(\bigwedge^\bullet L^*)$.

(iii) *Analogous results are obtained for L^* by replacing μ with γ.*

Proof. Let $s_1, s_2 \in \mathcal{P}^{(1,0)} = \Gamma^\infty(L)$. Using the bigrading properties we get

$$[s_1, s_2]_\Theta = \{\{s_1, \Theta\}_{\mathcal{R}}, s_2\}_{\mathcal{R}} = \{\{s_1, \psi\}_{\mathcal{R}}, s_2\}_{\mathcal{R}} + \{\{s_1, \mu\}_{\mathcal{R}}, s_2\}_{\mathcal{R}}$$

with $\{\{s_1, \psi\}_{\mathcal{R}}, s_2\}_{\mathcal{R}} \in \mathcal{P}^{(0,1)} = \Gamma^\infty(L^*)$ and $\{\{s_1, \mu\}_{\mathcal{R}}, s_2\}_{\mathcal{R}} \in \mathcal{P}^{(1,0)} = \Gamma^\infty(L)$. Thus $\mathrm{pr}_L([s_1, s_2]_\Theta) = [s_1, s_2]_\mu = \{\{s_1, \mu\}_{\mathcal{R}}, s_2\}_{\mathcal{R}}$. Analogously, we obtain (3.22). A standard computation finally shows that the extension of $[\cdot, \cdot]_\mu$ to multivector fields is given by (3.23) since (3.23) satisfies the same type of graded Leibniz rule and coincides with the Schouten–Nijenhuis bracket on the local generators. As one can see by counting degrees we have a well-defined map $\{\mu, \cdot\}_{\mathcal{R}} : \Gamma^\infty(\bigwedge^k L^*) \longrightarrow \Gamma^\infty(\bigwedge^{k+1} L^*)$. Thanks to the graded Leibniz rule for the Rothstein–Poisson bracket, this map is a graded derivation of the $\wedge$-product. A straightforward computation then shows that $i_s\{\mu, f\}_{\mathcal{R}} = i_s \mathrm{d}_L f$ and $i_{s_2} i_{s_1}\{\mu, \alpha\}_{\mathcal{R}} = i_{s_2} i_{s_1} \mathrm{d}_L \alpha$ for all $s, s_1, s_2 \in \Gamma^\infty(L)$, $f \in C^\infty(M)$ and $\alpha \in \Gamma^\infty(L^*)$. By the derivation property, $\{\mu, \cdot\}_{\mathcal{R}}$ coincides with d_L on the whole space $\Gamma^\infty(\bigwedge^\bullet L^*)$. The last statement follows analogously. ■

Recall that a Lie quasi-bialgebroid is a Lie algebroid $(A, [\,\cdot\,,\cdot\,]_A, \rho_A)$ together with a graded derivation d_{A^*} of degree one of $\Gamma^\infty\left(\bigwedge^\bullet A\right)$ with respect to both the $\wedge$-product and the Schouten–Nijenhuis bracket, and a 3-vector $\phi \in \Gamma^\infty(\bigwedge^3 A)$ such that $\mathrm{d}_{A^*}\phi = 0$ and $\mathrm{d}_{A^*}^2 = -[\phi, \cdot]_A$, see e.g. [23]. It is well-known that a graded derivation of degree one of $\Gamma^\infty\left(\bigwedge^\bullet A\right)$ defines a Schouten–Nijenhuis bracket on $\Gamma^\infty\left(\bigwedge^\bullet A^*\right)$. Thus a Lie quasi-bialgebroid is a pair (A, A^*), where A is a Lie algebroid and A^* is a quasi-Lie algebroid, such that the differential d_{A^*} of A^* is a graded derivation of the Schouten–Nijenhuis bracket on A and $\mathrm{d}_{A^*}^2 = -[\phi, \cdot]_A$ for some 3-vector $\phi \in \Gamma^\infty(\bigwedge^3 A)$ with $\mathrm{d}_{A^*}\phi = 0$, see e.g. [31]. A Lie bialgebroid [26] is obtained in the case that $\mathrm{d}_{A^*}^2 = 0$. Combining Lemmas 3.6 and 3.7 we obtain the following well-known result [29]:

Lemma 3.8. *Let $\Theta = \psi + \mu + \gamma + \phi \in \mathcal{P} \subset \Gamma^\infty\left(\bigwedge^\bullet \tau^\#(L \oplus L^*)\right)$ be as in Lemma 3.7 and assume now in addition $\{\Theta, \Theta\}_{\mathcal{R}} = 0$. If $\psi = 0$ then L is a Lie quasi-bialgebroid. If $\phi = 0$ then L^* is a Lie quasi-bialgebroid.*

3.4. Courant algebroids with Dirac structures

We shall now consider the case of a Courant algebroid $E = L \oplus L^*$ over M such that L is a Dirac structure. As we will see later in Corollary 4.4 a Courant algebroid E with a Dirac structure L is always of this form.

The element Θ from Theorem 3.5 now is given as a sum $\Theta = \psi + \mu + \gamma + \phi$ according to the bigrading. We split the tensor field T from Lemma 3.3 and $T^\sharp$, respectively, into their L and L^* components. Note also that we can identify T and $T^\sharp$ canonically, since $L \oplus L^*$ is canonically 'self-dual'. As before we set $\rho^i = \mathrm{d}x^i \circ \rho$ and define $r_i \in \Gamma^\infty(\bigwedge^\bullet \tau^\sharp(L \oplus L^*))$ for $i = 1, \ldots, n$ by $r_i = p_i - \Gamma^\beta_{i\alpha} a^\alpha \wedge a_\beta$, see Proposition A.11. Analogously, the anchor ρ splits into the two restrictions ρ_L and ρ_{L^*} to L and L^*, respectively. Then we have locally

$$\mathcal{J}(\rho_L^\sharp) = p_i \rho^i(a_\alpha) a^\alpha \quad \text{and} \quad \mathcal{J}(\rho_{L^*}^\sharp) = p_i \rho^i(a^\alpha) a_\alpha. \tag{3.25}$$

The above splitting of T and ρ into the components according to $E = L \oplus L^*$ now gives the splitting of Θ into the elements μ, γ, and ϕ. To identify these components, we *define* the global tensor fields

$$\mu = -\mathcal{J}(\rho_L^\sharp) + T|_{\bigwedge^2 L \otimes L^*} \tag{3.26}$$

$$\gamma = -\mathcal{J}(\rho_{L^*}^\sharp) + T|_{L \otimes \bigwedge^2 L^*} \tag{3.27}$$

$$\phi = T|_{\bigwedge^3 L^*}. \tag{3.28}$$

Because L is a Dirac structure one has $T|_{\bigwedge^3 L} = 0$ and therefore

$$\Theta = \mu + \gamma + \phi. \tag{3.29}$$

A little computation shows that $T|_{\bigwedge^2 L \otimes L^*}$ is three times the torsion [8] for the Lie algebroid L and analogously $T|_{L \otimes \bigwedge^2 L^*}$ is three times the torsion for the quasi Lie algebroid L^*. We further have

$$\phi(\sigma_1, \sigma_2, \sigma_3) = -\langle [\sigma_1, \sigma_2]_{\mathcal{C}}, \sigma_3 \rangle \tag{3.30}$$

for $\sigma_1, \sigma_2, \sigma_3 \in \mathcal{P}^{(0,1)} = \Gamma^\infty(L^*)$.

Let us look now at the local expressions. Let $x^1, \ldots, x^n$ be coordinates on M, $a_1, \ldots, a_k$ a local basis of sections of L and $a^1, \ldots, a^k$ the dual local basis of sections of L^*. We define local functions

$$c^\gamma_{\alpha\beta} = \langle [a_\alpha, a_\beta]_{\mathcal{C}}, a^\gamma \rangle \quad \text{and} \quad c^{\alpha\beta}_\gamma = \langle [a^\alpha, a^\beta]_{\mathcal{C}}, a_\gamma \rangle. \tag{3.31}$$

Furthermore, we have

$$\phi = \frac{1}{6} \phi^{\alpha\beta\gamma} a_\alpha \wedge a_\beta \wedge a_\gamma \quad \text{with } \phi^{\alpha\beta\gamma} = -(\langle [a^\alpha, a^\beta]_{\mathcal{C}}, a^\gamma \rangle). \tag{3.32}$$

Since we assume L to be a Dirac structure, all other combinations of structure functions as in Section 3.2 are either zero, or can be computed from the ones in (3.31) and (3.32) by using the properties of the Courant bracket $[\cdot, \cdot]_{\mathcal{C}}$.

Let $q^1, \ldots, q^n, p_1, \ldots, p_n$ be the induced coordinates on T^*M and let $\Gamma^\alpha_{i\beta}$ be the Christoffel symbols for the connection ∇ on L. We define

$$T^\gamma_{\alpha\beta} = T(a_\alpha, a_\beta, a^\gamma) = \rho^i(a_\alpha)\Gamma^\gamma_{i\beta} - \rho(a_\beta)^i \Gamma^\gamma_{i\alpha} - c^\gamma_{\alpha\beta} \tag{3.33}$$

$$T^{\alpha\beta}_\gamma = T(a^\alpha, a^\beta, a_\gamma) = \rho^i(a^\beta)\Gamma^\alpha_{i\gamma} - \rho^i(a^\alpha)\Gamma^\beta_{i\gamma} - c^{\alpha\beta}_\gamma \tag{3.34}$$

and then we have locally

$$T = \frac{1}{2} T^\gamma_{\alpha\beta} a^\alpha \wedge a^\beta \wedge a_\gamma + \frac{1}{2} T^{\alpha\beta}_\gamma a_\alpha \wedge a_\beta \wedge a^\gamma + \frac{1}{6} \phi^{\alpha\beta\gamma} a_\alpha \wedge a_\beta \wedge a_\gamma. \tag{3.35}$$

From this we immediately obtain the following statement:

Lemma 3.9. *Locally, μ and γ are given by*

$$\mu = -p_i \rho^i(a_\alpha) a^\alpha + \frac{1}{2} T^\gamma_{\alpha\beta} a^\alpha \wedge a^\beta \wedge a_\gamma \tag{3.36}$$

$$\gamma = -p_i \rho^i(a^\alpha) a_\alpha + \frac{1}{2} T^{\alpha\beta}_\gamma a_\alpha \wedge a_\beta \wedge a^\gamma. \tag{3.37}$$

In particular, $\mu \in \mathcal{P}^{(1,2)}$ and $\gamma \in \mathcal{P}^{(2,1)}$. In the local super-Darboux coordinates we have

$$\mu = -r_i \rho^i(a_\alpha) a^\alpha - \frac{1}{2} c^\gamma_{\alpha\beta} a^\alpha \wedge a^\beta \wedge a_\gamma \tag{3.38}$$

$$\gamma = -r_i \rho^i(a^\alpha) a_\alpha - \frac{1}{2} c^{\alpha\beta}_\gamma a_\alpha \wedge a_\beta \wedge a^\gamma. \tag{3.39}$$

Using the splitting in (3.38) and (3.39) we can compare with Roytenberg's expressions in [29, Eqs. (3.10) and (3.11)]. Note however, that this splitting depends on the choice of coordinates while (3.36) and (3.37) have intrinsic geometric meanings.

Corollary 3.10. *We have $\{\Theta, \Theta\}_{\mathcal{R}} = 0$, or equivalently*

$$\{\mu, \mu\}_{\mathcal{R}} = 0 \tag{3.40}$$

$$\frac{1}{2}\{\gamma, \gamma\}_{\mathcal{R}} + \{\mu, \phi\}_{\mathcal{R}} = 0 \tag{3.41}$$

$$\{\mu, \gamma\}_{\mathcal{R}} = 0 \tag{3.42}$$

$$\{\gamma, \phi\}_{\mathcal{R}} = 0. \tag{3.43}$$

Example 3.11. Let $E = TM \oplus T^*M$ be the standard Courant algebroid over M. Let ∇ be any torsion-free connection and construct the Rothstein–Poisson bracket on $\Gamma^\infty(\bigwedge^\bullet \tau^\# E)$. First we get

$$\gamma = 0, \quad \phi = 0, \quad \text{and} \quad \psi = 0. \tag{3.44}$$

For the only nontrivial element μ we find locally $\mu = -p_i \tau^\# \mathrm{d}x^i \in \Gamma^\infty(\tau^\# T^*M)$. The pulled back bundle $\tau^\# T^*M$ can be identified with the *annihilator subbundle* $\mathrm{Ver}(T^*M)^{\mathrm{ann}} \subseteq T^*(T^*M)$ of the *vertical* subbundle $\mathrm{Ver}(T^*M) \subseteq T(T^*M)$ in the usual way. This canonical identification allows us to identify $\tau^\# \mathrm{d}x^i$ with $\tau^* \mathrm{d}x^i = \mathrm{d}q^i$. Hence, under this identification, μ coincides with the *canonical one-form* $-\theta_0$ on T^*M.

We have some more corollaries to Corollary 3.10 and Lemma 3.8. First we note [29]:

Corollary 3.12. *On L we have given the structure of a quasi-Lie bialgebroid.*

Remark 3.13. If in addition $\{\mu, \phi\}_{\mathcal{R}} = 0$ is satisfied then (L, L^*) is a Lie bialgebroid [21]. But only if $\phi = 0$ the space of sections $\Gamma^\infty(L^*)$ is closed under the Courant bracket and L^* is a Dirac structure.

Given a Dirac structure L in a Courant algebroid E we always can find a maximal isotropic subbundle L' complementary to L and identify E with $L \oplus L^*$, see e.g. Corollary 4.4. Thus we have [29]:

Corollary 3.14. *A Courant algebroid E with a Dirac structure L is isomorphic to the double of the Lie quasi-bialgebroid $L \oplus L^*$.*

As shown in Lemma 3.7 the derived bracket $[\cdot, \cdot]_\mu$ is the Schouten–Nijenhuis bracket for the Lie algebroid structure on L given by the restriction of the Courant bracket and the anchor to $\Gamma^\infty(L)$. Further $[\cdot, \cdot]_\gamma$ defines a quasi-Lie algebroid structure on L^* where the bracket is given by

$$[\sigma_1, \sigma_2]_{L^*} = [\sigma_1, \sigma_2]_\gamma = \mathrm{pr}_{L^*}([\sigma_1, \sigma_2]_{\mathcal{C}}) \tag{3.45}$$

and the anchor by $\rho_{L^*} = \rho|_{L^*}$. The differential d_L is a graded derivation for the bracket $[\cdot, \cdot]_\gamma$ and the differential d_{L^*} is a graded derivation for the bracket $[\cdot, \cdot]_\mu$.

4. Smooth and formal deformations of Dirac structures

In this section we shall now establish the smooth and the formal deformation theory of Dirac structures. In the following E is a Courant algebroid with a fiber metric of signature zero and $L \subseteq E$ a Dirac structure as before.

4.1. Definition of smooth deformations

As motivation we first recall the well-known situation for Poisson manifolds, see e.g. [5, Sect. 18.5]: A *smooth deformation* π_t of a Poisson structure π_0 on M is a smooth map

$$\pi : I \times M \longrightarrow \bigwedge^2 TM \tag{4.1}$$

with $\pi_t = \pi(t, \cdot) \in \Gamma^\infty(\bigwedge^2 TM)$ for all $t \in I$ and $\pi(0, \cdot) = \pi_0$, such that

$$[\pi_t, \pi_t] = 0 \tag{4.2}$$

for all $t \in I$, where $I \subseteq \mathbb{R}$ is an open interval around zero. *Formal* deformations then are given by formal power series $\pi_t = \pi_0 + t\pi_1 + \cdots \in \Gamma^\infty(\bigwedge^2 TM)[[t]]$ such that $[\pi_t, \pi_t] = 0$ order by order in the formal parameter. A similar approach is possible in the case of symplectic manifolds.

Consider now a Dirac structure L in E. One possibility to define a *smooth* deformation of L is given by specifying a family of subbundles in terms of a family of projections. This way, we can encode the desired smoothness easily:

Definition 4.1. Let $L \subseteq E$ be a Dirac structure and let $I \subseteq \mathbb{R}$ be an open interval around zero. A smooth deformation of $L = L_0$ is a family of Dirac structures L_t with $t \in I$ such that there exists a smooth map

$$P : I \times M \longrightarrow \mathsf{End}(E) \tag{4.3}$$

with

(i) $P(t, m) : E_m \longrightarrow E_m$ for all $t \in I$ and $m \in M$
(ii) $P(t, m)^2 = P(t, m)$ for all $t \in I$ and $m \in M$
(iii) $\operatorname{Im} P_t = L_t$ for all $t \in I$, where $P_t = P(t, \cdot) \in \Gamma^\infty(\mathsf{End}(E))$.

Remark 4.2. Consider the pull-back bundle $\mathrm{pr}^\# E$, where $\mathrm{pr} : I \times M \longrightarrow M$ is the projection. Equivalent to the definition above we can consider a smooth deformation of L as a smooth subbundle $\mathfrak{L} \subseteq \mathrm{pr}^\# E$ such that every $L_t = \mathfrak{L}|_{\{t\}\times M} \subset E$ is a Dirac structure where $L_0 = L$.

While the above definition is conceptually clear and easy, it is not very suited for concrete computations. Thus we shall re-formulate the definition using additional geometric structures in Section 4.3. We also have to discuss the possible notions of equivalence in detail. However, we first recall two general well-known properties of the subbundles in question:

Theorem 4.3. *Let E be a vector bundle with a fiber metric $(\cdot, \cdot)$. Then there exits a positive definite fiber metric g and an isometry $J : E \longrightarrow E$ of $(\cdot, \cdot)$ with $J^2 = \mathsf{id}$, such that*

$$g(e_1, e_2) = (e_1, Je_2) \tag{4.4}$$

for all $e_1, e_2 \in \Gamma^\infty(E)$.

Proof. For the reader's convenience we sketch the proof: Choose a positive definite fiber metric k and define $A \in \Gamma^\infty(\mathsf{End}(E))$ by $k(Ae_1, e_2) = (e_1, e_2)$. Since A turns out to be k-symmetric we can use its polar decomposition $A = \sqrt{A^2}J$. Then $g(e_1, e_2) = (e_1, Je_2)$ has the required properties. ∎

Corollary 4.4. *Let E be a vector bundle with even fiber dimension $2k$ and let $(\cdot, \cdot)$ be a bilinear form on E of signature zero. Let further L be a maximal isotropic subbundle of E. Choose g and J according to Theorem 4.3. Then*

$$E = L \oplus J(L) \quad \textit{and} \quad L^{\perp_g} = J(L) \cong L^*. \tag{4.5}$$

Theorem 4.5. *Let E be a vector bundle, $I \subset \mathbb{R}$ an open interval around zero and let L_t for $t \in I$ be a smooth family of subbundles of E. Then there exits a vector bundle automorphism U_t of E over the identity $\mathsf{id} : M \longrightarrow M$, smoothly depending on $t \in I$ such that*

$$L_t = U_t(L_0). \tag{4.6}$$

If E is a Courant algebroid and L_t a family of maximal isotropic subbundles, then we can also achieve that U_t is an isometry of the symmetric bilinear form $h = (\cdot, \cdot)$ for all $t \in I$.

Proof. The theorem can be proved along the lines of [12, Lem. 1.1.5]. ■

4.2. The problem of equivalence

Let L be a Dirac structure in a Courant algebroid E and L_t a smooth deformation of $L = L_0$. We know from Theorem 4.5 that there exists an isometry U_t of E smoothly depending on t such that $L_t = U_t(L_0)$. Thus, in this general concept it seems natural to define a *trivial deformation* as a deformation L_t such that we can find a time dependent U_t which is not only an isometry but also a *Courant algebroid automorphism*. If we further ask whether two smooth deformations L_t and L'_t are equivalent, one is tempted to require the existence of a time dependent Courant algebroid automorphism U_t, such that $L'_t = U_t(L_t)$.

However, in the case of of the standard Courant algebroid $TM \oplus T^*M$, due to Proposition 2.8, this would mean that we have the gauge transformations by closed two-forms as equivalence transformations. But then every two Dirac structures given by presymplectic forms would be equivalent. Hence we see that in the case of $TM \oplus T^*M$ we can not permit every Courant algebroid automorphism as an equivalence transformation as long as we want to reproduce the common results for the deformation theory of symplectic forms.

In the case of $E = TM \oplus T^*M$, we know that every automorphism is given by the product of a gauge transformation and a lifted diffeomorphism $\mathcal{F}\phi$. As we do not want gauge transformations as equivalence transformations we have to consider the *lifted diffeomorphisms*. Indeed, given a presymplectic form ω on a manifold M and a diffeomorphism ϕ of M, one can easily show [4] that the equation

$$\mathcal{B}\phi(\operatorname{graph}\omega) = \operatorname{graph}(\phi^*\omega) \tag{4.7}$$

is satisfied. Analogously we have

$$\mathcal{B}\phi(\operatorname{graph}\pi) = \operatorname{graph}(\phi^*\pi) \tag{4.8}$$

for a Poisson tensor π on M. This motivates the following definition of equivalent deformations of Dirac structures which reduce to the well-known situation in the Poisson or symplectic case:

Definition 4.6. Let $L \subset TM \oplus T^*M$ be a Dirac structure in the standard Courant algebroid. Two smooth deformations L_t and L'_t of L are called equivalent, if there exists a smooth curve of diffeomorphisms ϕ_t of M such that $L'_t = \mathcal{F}\phi_t(L_t)$. A smooth deformation is called trivial, if there exists a smooth curve of diffeomorphisms ϕ_t such that $L_t = \mathcal{F}\phi_t(L_0)$.

While for the standard Courant algebroid this seems to be the reasonable definition of equivalent deformations, in general it will be more difficult: for any vector bundle $E \longrightarrow M$ we have the exact sequence of groups

$$1 \longrightarrow \operatorname{Gau}(E) \longrightarrow \operatorname{Aut}(E) \longrightarrow \operatorname{Diffeo}(M) \longrightarrow 1, \tag{4.9}$$

where $\operatorname{Gau}(E)$ denotes those vector bundle automorphisms of E which induce the identity on M, and the last arrow assigns to an arbitrary vector bundle automorphism $\Phi : E \longrightarrow E$ the induced diffeomorphism ϕ of M. However, quite unlike for the Courant algebroid $TM \oplus T^*M$, in general this exact sequence does *not* split. Furthermore, even if the sequence splits, it is not clear, whether the split can be chosen in a reasonable way. In fact, if E is associated to the frame-bundle, then one can choose a splitting.

Since it is precisely this canonical splitting in the case of $TM \oplus T^*M$ which we use for Definition 4.6 there seems to be no simple way out. One possibility would be the following: since we are only interested in smooth curves of diffeomorphisms ϕ_t of M with $\phi_0 = \mathsf{id}_M$ we know that such a curve is the time evolution of a time-dependent vector field X_t on M. After the *choice* of a connection ∇ on E we can lift X_t horizontally to E and consider its time evolution Φ_t on E. Then we can use Φ_t instead of $\mathcal{F}\phi_t$ to formulate a definition of equivalence and trivial deformations analogously to Definition 4.6. However, this would depend explicitly on the choice of a connection. We shall come back to this problem in a future work. At the present stage, the question of equivalence of smooth deformations of Dirac structures in a general Courant algebroid has to be left unanswered.

4.3. Rewriting the deformation problem

To study the formal deformation theory of Dirac structures we first have to think about an appropriate description for such deformations. Given a Courant algebroid E with Dirac structure L we *choose* an isotropic complement L' to L (for example with the help of Corollary 4.4) and identify L' with L^*. Then we can write $E = L \oplus L^*$, where the fiber metric on E translates to the natural pairing on $L \oplus L^*$. Thus we may assume that E has this form in the following. Note however, that we still have to discuss the influence of this chosen isomorphism later.

Locally a small deformation L_t of L could be understood as the graph of a map $\omega_t : L \longrightarrow L^*$. Indeed, over a compact subset $K \subseteq M$ a smooth deformation L_t can be written as the graph of some ω_t provided t is sufficiently small. Globally in M, this needs not to be true whence smooth deformation theory becomes highly non-trivial. However, since we will mainly be interested in formal deformations (to be thought of as formal Taylor expansions of smooth deformations) the idea of looking at graphs will be sufficient for us. The claim that L_t is isotropic allows us to identify ω_t with a 2-form in L. To ensure that $\Gamma^\infty(L_t)$ is closed under the Courant bracket and therefore is a Dirac structure leads to an additional requirement for ω_t.

In the following considerations we will first omit the dependency on t. So let $\omega \in \Omega^2(L)$ be a 2-form. Then $\mathrm{graph}(\omega)$ is integrable, i.e. closed under the Courant bracket, if and only if for all $s_1, s_2, s_3 \in \Gamma^\infty(L)$

$$\begin{aligned} 0 &= \langle [s_1 + \omega(s_1), s_2 + \omega(s_2)]_{\mathcal{C}}, s_3 + \omega(s_3)\rangle \\ &= \langle [s_1, \omega(s_2)]_{\mathcal{C}}, s_3\rangle + \langle [\omega(s_1), s_2]_{\mathcal{C}}, s_3\rangle + \langle [s_1, s_2]_{\mathcal{C}}, \omega(s_3)\rangle \\ &\quad + \langle [s_1, \omega(s_2)]_{\mathcal{C}}, \omega(s_3)\rangle + \langle [\omega(s_1), s_2]_{\mathcal{C}}, \omega(s_3)\rangle + \langle [\omega(s_1), \omega(s_2)]_{\mathcal{C}}, s_3\rangle \\ &\quad + \langle [\omega(s_1), \omega(s_2)]_{\mathcal{C}}, \omega(s_3)\rangle. \end{aligned} \tag{4.10}$$

The constant term in ω vanishes as L is assumed to be a Dirac structure throughout. Moreover, this equation combines linear, quadratic and cubic terms in ω. In order to analyze this equation in more detail, we use the Rothstein–Poisson bracket.

Lemma 4.7. *Let $E = L \oplus L^*$ be a Courant algebroid with L a Dirac structure and let $\omega \in \Gamma^\infty(\bigwedge^2 L^*)$ be a 2-form. Then* $\mathrm{graph}(\omega) \subseteq E$ *is a Dirac structure if and only if*

$$\{\mu, \omega\}_{\mathcal{R}} + \frac{1}{2}\{\{\omega, \gamma\}_{\mathcal{R}}, \omega\}_{\mathcal{R}} + \frac{1}{6}\{\{\{\phi, \omega\}_{\mathcal{R}}, \omega\}_{\mathcal{R}}, \omega\}_{\mathcal{R}} = 0. \tag{4.11}$$

Proof. Replacing all Courant brackets by the derived bracket using Θ gives (4.11) after a straightforward computation. ■

Due to the bigrading properties of the Rothstein–Poisson bracket the definition

$$[\eta_1, \eta_2, \eta_3]_\phi = \{\{\{\phi, \eta_1\}_{\mathcal{R}}, \eta_2\}_{\mathcal{R}}, \eta_3\}_{\mathcal{R}} \tag{4.12}$$

gives a well-defined trilinear map

$$\Gamma^\infty\left(\bigwedge\nolimits^k L^*\right) \times \Gamma^\infty\left(\bigwedge\nolimits^l L^*\right) \times \Gamma^\infty\left(\bigwedge\nolimits^m L^*\right) \longrightarrow \Gamma^\infty\left(\bigwedge\nolimits^{k+l+m-3} L^*\right). \tag{4.13}$$

Moreover, because ϕ is a pull-back section this map is independent of the connection used for constructing the Rothstein–Poisson bracket. With the definitions from Lemma 3.7 we can write (4.11) equivalently as

$$\mathrm{d}_L\omega + \frac{1}{2}[\omega, \omega]_\gamma + \frac{1}{6}[\omega, \omega, \omega]_\phi = 0. \tag{4.14}$$

This is the fundamental equation for ω which has been derived by Roytenberg in his approach in another context, see [31].

Eq. (4.14) is precisely the sorting of (4.10) by the homogeneous monomials in ω and hence independent of the usage of the Rothstein–Poisson bracket. Nevertheless, we can use the Rothstein–Poisson bracket to obtain algebraic identities for the three parts of (4.14) which are very hard to obtain without the Rothstein–Poisson bracket.

4.4. Formal deformations

Following the general idea of formal deformation theory, namely to solve a non-linear algebraic equation order by order in terms of formal power series [13,14], we consider solutions of (4.10) in the sense of formal power series. Since ω should be a 'small' deformation we make the Ansatz

$$\omega = t\omega_1 + t^2\omega_2 + \cdots = \sum_{t=1}^{\infty} t^r \omega_r \in t\Gamma^\infty \left(\bigwedge^2 L^*\right)[[t]], \tag{4.15}$$

where $\omega_1, \omega_2, \ldots$ have to be determined recursively. Since $\omega_r \in \Gamma^\infty(\bigwedge^2 L^*)$, we can interpret the deformation as a 2-cochain in the Lie algebroid complex of L, viewed only as a Lie algebroid. The following lemma is now crucial for the cohomological approach:

Lemma 4.8. *Let $\eta \in \Gamma^\infty(\Lambda^2 L^*)$ be a two-form. Then*

$$\mathrm{d}_\eta = \{\mu, \cdot\}_{\mathcal{R}} + \{\{\eta, \gamma\}_{\mathcal{R}}, \cdot\}_{\mathcal{R}} + \frac{1}{2}\{\{\{\phi, \eta\}_{\mathcal{R}}, \eta\}_{\mathcal{R}}, \cdot\}_{\mathcal{R}} = \mathrm{d}_L + [\eta, \cdot]_\gamma + \frac{1}{2}[\eta, \eta, \cdot]_\phi \tag{4.16}$$

defines a graded derivation of degree one of the $\wedge$-product such that

$$\mathrm{d}_\eta \left(\mathrm{d}_L \eta + \frac{1}{2}[\eta, \eta]_\gamma + \frac{1}{6}[\eta, \eta, \eta]_\phi\right) = 0. \tag{4.17}$$

Proof. Using the derived bracket formalism this is a straightforward computation. ■

The following theorem shows that the solvability of (4.10) or equivalently (4.14) order by order in the formal parameter leads to a cohomological obstruction in the usual way:

Theorem 4.9. *Let $E = L \oplus L^*$ be a Courant algebroid with a Dirac structure L and let $\omega_t = t\omega_1 + t^2\omega_2 + \cdots + t^N\omega_N \in \Gamma^\infty(\bigwedge^2 L^*)[[t]]$ be a formal deformation of L of order N, i.e. the equation*

$$\mathrm{d}_L\omega_t + \frac{1}{2}[\omega_t, \omega_t]_\gamma + \frac{1}{6}[\omega_t, \omega_t, \omega_t]_\phi = 0 \tag{4.18}$$

is satisfied up to order N. Then

$$R_{N+1} = -\frac{1}{2}\sum_{i=1}^{N}[\omega_i, \omega_{N+1-i}]_\gamma - \frac{1}{6}\sum_{i+j+k=N+1}[\omega_i, \omega_j, \omega_k]_\phi \in \Gamma^\infty\left(\bigwedge^3 L^*\right) \tag{4.19}$$

is closed with respect to d_L, and ω_t can be extended to a deformation of order $N+1$ if and only if R_{N+1} is exact.

Proof. The proof is essentially the usual argument of formal deformation theory. Let $\omega_{N+1} \in \Gamma^\infty(\bigwedge^2 L^*)$ be arbitrary and set $\omega_t' = \omega_t + t^{N+1}\omega_{N+1}$. Then

$$\begin{aligned}
&\mathrm{d}_L\omega_t' + \frac{1}{2}[\omega_t', \omega_t']_\gamma + \frac{1}{6}[\omega_t', \omega_t', \omega_t']_\phi \\
&\quad = t^{N+1}\left(\mathrm{d}_L\omega_{N+1} + \frac{1}{2}\sum_{i=1}^{N}[\omega_i, \omega_{N+1-i}]_\gamma + \frac{1}{6}\sum_{i+j+k=N+1}[\omega_i, \omega_j, \omega_k]_\phi\right) + o(t^{N+2}),
\end{aligned}$$

whence ω'_t satisfies (4.14) up to order $N+1$ if and only if $\mathrm{d}_L \omega_{N+1} = R_{N+1}$, i.e. R_{N+1} is exact with respect to d_L. On the other hand, R_{N+1} is always closed. Indeed, by Lemma 4.8 applied to ω'_t we get

$$0 = t^{N+1} \mathrm{d}_L \left(\frac{1}{2} \sum_{i=1}^{l} [\omega_i, \omega_{N+1-i}]_\gamma + \frac{1}{6} \sum_{i+j+k=N+1} [\omega_i, \omega_j, \omega_k]_\phi \right) + o(t^{N+2}),$$

which implies $\mathrm{d}_L R_{N+1} = 0$. ■

Remark 4.10. From the proof it is clear that the whole derived bracket formalism enters only in showing that R_{N+1} is d_L-closed. This is in some sense the nontrivial statement of the theorem. In principle, this can also be shown directly using only (4.10) and the algebraic identities for the Courant bracket. However, the computations are very much involved without using the nice derived bracket formalism. Nevertheless, it should be emphasized that the characterization of the order-by-order obstruction to solve (4.10) by the third Lie algebroid cohomology of the Dirac structure is *independent* of the choices we made in order to obtain the Rothstein–Poisson bracket.

4.5. Examples: Presymplectic and Poisson manifolds

Let us now discuss some examples in order to show that the deformation theory of Dirac structures generalizes the well-known deformation theories of presymplectic and Poisson structures.

Let (M, ω) be a presymplectic manifold, and consider the standard Courant algebroid $TM \oplus T^*M$ with the Dirac structure $L = \mathrm{graph}(\omega)$. In this case T^*M is a complement of L and we can identify $L^* \cong T^*M$. There is also a canonical identification of L with TM, which is given by the restriction of the gauge transformation $\tau_{-\omega}$ to L. Because ω is closed, $\tau_{-\omega}$ is a Courant algebroid automorphism and we have the identification $L \oplus L^* \cong TM \oplus T^*M$, where the Courant algebroid structure on the right hand side is still the standard one. Because $L^* \cong T^*M$ is a Dirac structure with trivial Lie algebroid structure, according to our theory a smooth deformation of $L \cong TM$ is given by a closed time-dependent two-form η_t with $\eta_0 = 0$. The deformation of the original Dirac structure is then given by $L_t = \mathrm{graph}(\omega + \eta_t)$, i.e. by the deformation of the presymplectic form ω. Thus we retrieve the common results in this case.

Usually in formal deformation theory, the infinitesimally inequivalent deformations are parameterized by a second cohomology relevant for the deformation problem while the third cohomology gives the obstructions for the existence of order-by-order deformations. In our case, one would expect the second Lie algebroid cohomology to be the relevant one.

For the usual deformation theory of symplectic forms or Poisson bivectors this is indeed the case. However, in the general case, the situation is more subtle. To see this, we consider the following example:

First recall that the Lie algebroid cohomology of a Dirac structure coming from a presymplectic structure coincides with the de Rham cohomology. Then, for a presymplectic manifold, two formal deformations ω_t and ω'_t of the presymplectic form ω_0 are equivalent iff there exists a formal diffeomorphism $\phi_t = \exp(\mathcal{L}_{X_t})$ with $X_t = tX_1 + \cdots \in t\Gamma^\infty(TM)[[t]]$, such that $\phi_t \omega_t = \omega'_t$. This is the reasonable definition of 'deformations up to formal diffeomorphisms'. In first order this equation reads as

$$\omega'_1 - \omega_1 = \mathcal{L}_{X_1} \omega_0 = \mathrm{d} i_{X_1} \omega_0. \tag{4.20}$$

If there is a $\alpha \in \Omega^1(M)$, such that $\mathrm{d}\alpha = \omega'_1 - \omega_1$, then we must find X_1 with $i_{X_1}\omega_0 = \alpha$. For a *symplectic* form ω_0 this is always possible, so nontrivial deformations only exist if $H^2_{\mathrm{dR}}(M)$ is nontrivial. However, if we start with a presymplectic form ω_0, there might be *no* X_1 such that $i_{X_1}\omega_0 = \alpha$ and the triviality of $H^2_{\mathrm{dR}}(M)$ is not sufficient for the rigidity of M as a presymplectic manifold. Because the presymplectic deformation is a special case of the deformation of Dirac structures, the obstructions for the existence of non-trivial deformations are not in the second Lie algebroid cohomology of L. This is probably the most surprising feature of the deformation theory of Dirac structures.

Remark 4.11. One might wonder whether this is just an artifact of our notion of equivalence based on formal diffeomorphisms. However, if one decides to use the notion of equivalence suggested by Theorem 4.5 (which we do not prefer, see the discussion in Section 4.2), then the situation is even worse: All deformations of presymplectic forms in this sense become equivalent, while the second Lie algebroid cohomology might be nontrivial.

Finally, let us consider a Poisson manifold (M, π), and consider the Dirac structure $L = \mathrm{graph}(\pi)$ in the standard Courant algebroid $TM \oplus T^*M$. We choose TM as complement to L so that we can identify L^* with TM. Observe that in this case $L^* \cong TM$ is again a Dirac structure but unlike as above the Lie algebroid structure on L^* is non-trivial. We further identify L with T^*M via $\rho^*|_L$. Hence, we have the identification $L \oplus L^* = T^*M \oplus TM$, but the Courant algebroid structure on the right hand side now is not the standard one. The differential d_L becomes the differential given by π, i.e. $\mathrm{d}_\pi = [\pi, \cdot\,]$, and the bracket on $L^* \cong TM$ is the canonical Schouten–Nijenhuis bracket. Deformations of L are given by time-dependent bivector fields λ_t such that

$$\mathrm{d}_\pi \lambda_t + \frac{1}{2}[\lambda_t, \lambda_t] = 0. \tag{4.21}$$

Thanks to $[\pi, \pi] = 0$, this equation is equivalent to

$$[\pi + \lambda_t, \pi + \lambda_t] = 0. \tag{4.22}$$

We conclude that deformations $\pi_t = \pi + \lambda_t$ of the Poisson tensor π are the same as deformations of the corresponding Dirac structure L.

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Appendix A. The Rothstein–Poisson bracket

A.1. Definition of the Rothstein–Poisson bracket

Let $F \longrightarrow N$ be a vector bundle over a symplectic manifold (N, ω) and let $\pi = -\omega^{-1}$ be the Poisson tensor for the induced Poisson bracket on N, i.e. $\{f, g\} = \pi(\mathrm{d}f, \mathrm{d}g)$. Further let h be a pseudo-riemannian metric on N and ∇ a metric connection. We denote local coordinates on N by $x^1, \ldots, x^n$, local basis sections of F by $s_1, \ldots, s_k$ and the dual sections of F^* by $s^1, \ldots, s^k$. With the local expression

$$\hat{R} = \frac{1}{2}\pi^{ij} h^{AB} R^C_{Ajk} \partial_i \otimes s_B \wedge s_C \otimes \mathrm{d}x^k \tag{A.1}$$

we get a well defined global section $\hat{R} \in \Gamma^\infty(TN \otimes \bigwedge^2 F \otimes T^*N)$, where h^{AB} and R^A_{Bij} are the local expressions for the pseudo-riemannian metric h^{-1} and the curvature R in coordinates. A section $S \in \Gamma^\infty(TN \otimes \bigwedge^k F \otimes T^*N)$ can be interpreted as a map

$$S : \Gamma^\infty\left(TN \otimes \bigwedge\nolimits^\bullet F\right) \longrightarrow \Gamma^\infty\left(TN \otimes \bigwedge\nolimits^{\bullet+k} F\right) \tag{A.2}$$

by

$$(X \otimes \phi \otimes \eta)(Y \otimes \psi) = \eta(Y) X \otimes \phi \wedge \psi. \tag{A.3}$$

We therefore can form powers of $\hat{R}$ by composition of maps. Because $\hat{R}$ increases the degree of the part in $\bigwedge^\bullet F$ by two, $\hat{R}$ is nilpotent and we have a well-defined section

$$(\mathrm{id} - \hat{R})^{-\frac{1}{2}} = \mathrm{id} + \frac{1}{2}\hat{R} + \frac{3}{8}\hat{R}^2 + \cdots, \tag{A.4}$$

where $\mathrm{id} = \partial_i \otimes 1 \otimes \mathrm{d}x^i$ is the identity map in $\Gamma^\infty(TN \otimes \bigwedge^\bullet F)$. For a section $S \in \Gamma^\infty(TN \otimes \bigwedge^\bullet F \otimes T^*N)$ we define the local section S^i_j of $\bigwedge^\bullet F$ by

$$S = \partial_i \otimes S^i_j \otimes \mathrm{d}x^j. \tag{A.5}$$

In the following we denote by $i(\sigma)\psi$ and $j(\sigma)\psi$ the interior product of a section $\sigma \in \Gamma^\infty(F^*)$ with an element $\psi \in \Gamma^\infty(\bigwedge^\bullet F)$ from the left and right, respectively.

Theorem A.1 (*Rothstein–Poisson Bracket [3,28]*). *Given a vector bundle F together with ∇, h and ω as above we have a super-Poisson bracket $\{\cdot,\cdot\}_{\mathcal{R}}$ on $\Gamma^\infty\left(\bigwedge^\bullet F\right)$, called the Rothstein–Poisson bracket, which is locally given by*

$$\{\phi,\psi\}_{\mathcal{R}} = \pi^{ij}\Big((1-\hat{R})^{-\frac{1}{2}}\Big)_i^k \wedge \Big((1-\hat{R})^{-\frac{1}{2}}\Big)_j^l \wedge \nabla_{\partial_k}\phi \wedge \nabla_{\partial_l}\psi + h_{AB}\, j(s^A)\phi \wedge i(s^B)\psi. \tag{A.6}$$

That $\{\cdot,\cdot\}_{\mathcal{R}}$ is a super-Poisson bracket means that for all $\phi \in \Gamma^\infty(\bigwedge^k F)$, $\psi \in \Gamma^\infty(\bigwedge^l F)$ and $\eta \in \Gamma^\infty(\bigwedge^\bullet F)$ we have

(i) $\{\phi,\psi\}_{\mathcal{R}} = -(-1)^{kl}\{\psi,\phi\}_{\mathcal{R}}$
(ii) $\{\phi,\psi\wedge\eta\}_{\mathcal{R}} = \{\phi,\psi\}_{\mathcal{R}}\wedge\eta + (-1)^{kl}\psi\wedge\{\phi,\eta\}_{\mathcal{R}}$
(iii) $\{\phi,\{\psi,\eta\}_{\mathcal{R}}\}_{\mathcal{R}} = \{\{\phi,\psi\}_{\mathcal{R}},\eta\}_{\mathcal{R}} + (-1)^{kl}\{\psi,\{\phi,\eta\}_{\mathcal{R}}\}_{\mathcal{R}}$.

A.2. The Rothstein bracket for pullback bundles

Let $\pi_M : E \longrightarrow M$ be a vector bundle, N a manifold and let $f : N \longrightarrow M$ be a smooth map. We denote by $f^\#\pi_M : f^\# E \longrightarrow N$ the pullback bundle with respect to f. Given a section $e \in \Gamma^\infty(E)$ the pullback section $f^\# e \in \Gamma^\infty(f^\# E)$ is then defined by

$$f^\# e = e \circ f. \tag{A.7}$$

A local basis $u_1,\ldots,u_K \in \Gamma^\infty(E|_U)$ of E defined on some open set $U \subseteq M$ leads to a local basis $f^\# u_1,\ldots,f^\# u_K$ of $f^\# E$ defined on $f^{-1}(U) \subseteq N$. Given a connection ∇ on E we have the induced connection $f^\#\nabla$ on $f^\# E$, where for pullback sections $f^\# e$ with $e \in \Gamma^\infty(E)$ we have for $Y \in \Gamma^\infty(TN)$

$$f^\#\nabla_Y(f^\# e) = f^\#(\nabla_{Tf(Y)} e). \tag{A.8}$$

In the following we will look at the case $N = T^*M$ with $f = \tau : T^*M \longrightarrow M$ the cotangent projection. Let $x^1,\ldots,x^n$ be coordinates on $U \subseteq M$ and $q^1,\ldots,q^n, p_1,\ldots,p_n$ the induced bundle coordinates on T^*U. The sign of the canonical Poisson bracket on T^*M is choosen such that $\{q^i, p_j\} = \delta^i_j$. Observe that for pullback sections $\tau^\# u \in \tau^\#(\Gamma^\infty(E))$ we have

$$(\tau^\#\nabla)_{\frac{\partial}{\partial q^i}}\tau^\# u = \tau^\#(\nabla_{\frac{\partial}{\partial x^i}} u) \quad \text{and} \quad (\tau^\#\nabla)_{\frac{\partial}{\partial p_i}}\tau^\# u = 0. \tag{A.9}$$

In particular, for a general section $s \in \Gamma^\infty(\tau^\# E)$ the expression $(\tau^\#\nabla)_{\frac{\partial}{\partial p_i}} s$ is independent of the connection ∇ and therefore we set

$$\frac{\partial s}{\partial p_i} = (\tau^\#\nabla)_{\frac{\partial}{\partial p_i}} s \tag{A.10}$$

for the covariant derivative of s with respect to $\frac{\partial}{\partial p_i}$.

Lemma A.2. *Let $\pi : E \longrightarrow M$ be a vector bundle with connection ∇^E and a fiber metric h. Look at the pullback bundle $F = \tau^\# E$ over the symplectic manifold T^*M together with the pullback connection $\nabla^F = \tau^\#\nabla^E$ and the pullback metric $\tau^* h$. Then the map $\hat{R}^F$ as defined in (A.1) satisfies*

$$\hat{R}^F\left(\frac{\partial}{\partial q^i}\otimes\psi\right) = \frac{1}{2}\frac{\partial}{\partial p_j}\otimes\tau^\#\left(h^{AB}\Big(R^E\Big)^C_{Aij} u_B\wedge u_C\right)\wedge\psi$$

$$\hat{R}^F\left(\frac{\partial}{\partial p_i}\otimes\psi\right) = 0,$$

where $(R^E)^B_{Aij}$ is the curvature of ∇^E with respect to the appropriate coordinates.

Proof. This is a straightforward computation. ■

With this lemma it follows immediately that $(\hat{R}^F)^k = 0$ for $k \geq 2$ and therefore

$$(\mathrm{id} - \hat{R}^F)^{-\frac{1}{2}} = \mathrm{id} + \frac{1}{2}\hat{R}^F. \tag{A.11}$$

Hence in this case we get a more explicit formula for the Rothstein–Poisson bracket.

Lemma A.3. *With the above definitions the Rothstein–Poisson bracket on $\Gamma^\infty(\tau^\#(\bigwedge^\bullet E))$ is given by*

$$\begin{aligned}\{\phi, \psi\}_{\mathcal{R}} = {} & \nabla^F_{\frac{\partial}{\partial q^i}}\phi \wedge \frac{\partial}{\partial p_i}\psi - \frac{\partial}{\partial p_i}\phi \wedge \nabla^F_{\frac{\partial}{\partial q^i}}\psi - \frac{1}{2}\tau^\#\left(h^{AB}\left(R^E\right)^C_{Aij} u_B \wedge u_C\right) \wedge \frac{\partial}{\partial p_i}\phi \wedge \frac{\partial}{\partial p_j}\psi \\ & + \tau^* h_{AB}\; j(\tau^\# u^A)\phi \wedge i(\tau^\# u^B)\psi.\end{aligned} \tag{A.12}$$

A.3. Super-Darboux coordinates

Let us choose a local basis of sections $s_1, \ldots, s_k$ of the bundle E such that the functions $h_{AB} = h(s_A, s_B)$ are constant. If we calculate the Rothstein–Poisson bracket for the coordinate functions q^i, p_j and the local sections $\tau^\# u_A$ of $\tau^\# E$ we get the equations

$$\begin{aligned} & \{q^i, q^j\}_{\mathcal{R}} = 0 \quad \{q^i, p_j\}_{\mathcal{R}} = \delta^i_j \\ & \{p_i, p_j\}_{\mathcal{R}} = -\frac{1}{2}\tau^\#\left(h^{AB}\left(R^E\right)^C_{Aij} u_B \wedge u_C\right) \quad \text{and} \quad \{q^i, \tau^\# u_A\}_{\mathcal{R}} = 0 \\ & \{p_i, \tau^\# u_A\}_{\mathcal{R}} = -\tau^\#\left(\Gamma^B_{iA} u_B\right) \quad \text{and} \quad \{\tau^\# u_A, \tau^\# u_B\}_{\mathcal{R}} = \tau^* h_{AB}.\end{aligned} \tag{A.13}$$

We see that $C^\infty(T^*M)$ is in general not closed under the Rothstein–Poisson bracket.

Proposition A.4. *Let the local sections r_i of the bundle $\Gamma^\infty\left(\bigwedge^\bullet(\tau^\# E)\right)$ be defined by*

$$r_i = p_i - \frac{1}{2}\tau^\#\left(h^{AB}\,\Gamma^C_{iA}\, u_B \wedge u_C\right). \tag{A.14}$$

Then the following equations are satisfied:

$$\{q^i, r_j\}_{\mathcal{R}} = \delta^i_j \quad \textit{and} \quad \{\tau^\# u_A, \tau^\# u_B\}_{\mathcal{R}} = h_{AB}, \tag{A.15}$$

and

$$\{q^i, q^j\}_{\mathcal{R}} = \{q^i, \tau^\# u_A\}_{\mathcal{R}} = \{r_i, r_j\}_{\mathcal{R}} = \{r_i, \tau^\# u_A\}_{\mathcal{R}} = 0. \tag{A.16}$$

Proof. A direct calculation using the fact that the connection is metric leads to the result. ■

A.4. Grading for polynomial sections

Let $\mathcal{P} \subset \Gamma^\infty\left(\bigwedge^\bullet(\tau^\# E)\right)$ be the sections which are polynomial in the momenta, i.e. sections which locally can be written as a linear combination of local sections of the form

$$h^{A_1 \ldots A_s}\tau^\# u_{A_1} \wedge \cdots \wedge \tau^\# u_{A_s} \tag{A.17}$$

for $0 \leq s \leq k$ with $h_{A_1 \ldots A_s} \in \mathcal{P}^\bullet(T^*M)$ polynomial functions on T^*M. From (A.12) we get the following:

Lemma A.5. *The space $\mathcal{P}$ is closed under the Rothstein–Poisson bracket.*

Definition A.6. Let the map $\deg : \mathcal{P} \longrightarrow \mathcal{P}$ be defined by the local formula

$$\deg = 2p_i \frac{\partial}{\partial p_i} + \tau^\# u_A \wedge i(\tau^\# u^A). \tag{A.18}$$

For an element $\phi \in \mathcal{P}$ we say that ϕ is of degree r if the equation $\deg \phi = r\phi$ is satisfied. We denote the set of all such elements by $\mathcal{P}^r$.

Remark A.7. (i) Elements with degree zero can be identified with functions on M and elements with degree one with sections in E, i.e.

$$\mathcal{P}^0 = \tau^*(C^\infty(M)) \quad \text{and} \quad \mathcal{P}^1 = \tau^\#(\Gamma^\infty(E)). \tag{A.19}$$

(ii) The degree given by deg can be used to calculate the signs for the super-Poisson structure given by the Rothstein bracket because the momenta always count twice.

Lemma A.8. *The Rothstein–Poisson bracket is of degree -2 for the grading given by* deg.

A.5. The case $E = L \oplus L^*$

Let $L \longrightarrow M$ be a vector bundle with a connection ∇. We also have a connection on the dual bundle L^* and therefore a connection ∇^E on $E = L \oplus L^*$, which is metric with respect to the canonical bilinear form on $L \oplus L^*$, given by

$$\langle (s_1, \alpha_1), (s_2, \alpha_2) \rangle = \alpha_1(s_2) + \alpha_2(s_1) \tag{A.20}$$

for $s_1, s_2 \in \Gamma^\infty(L)$ and $\alpha_1, \alpha_2 \in \Gamma^\infty(L^*)$.

Let $x^1, \ldots, x^n$ be coordinates on M, $a_1, \ldots, a_k$ be a local basis of L and $a^1, \ldots, a^k$ be the dual basis of L^*. Let $R^\beta_{\alpha ij}$ be the curvature on L in coordinates. The curvature on L^* then is given in the dual coordinates by $-R^\beta_{\alpha ij}$. If we choose

$$(u_1, \ldots, u_A, \ldots, u_{2k}) = (a_1, \ldots, a_k, a^1, \ldots, a^k) \tag{A.21}$$

as a local basis of $L \oplus L^*$ we get for the curvature on $E = L \oplus L^*$

$$\left(R^E\right)^B_{Aij} = \begin{cases} R^B_{Aij} & \text{for } 1 \le A, B \le k \\ -R^{A-k}_{B-k,ij} & \text{for } k+1 \le A, B \le 2k \\ 0 & \text{otherwise.} \end{cases} \tag{A.22}$$

Now let $F = \tau^\#(L \oplus L^*) \longrightarrow T^*M$ again be the pullback bundle. Because of the special form of the curvature and the fiber metric in the given coordinates, we can simplify the formula for the Rothstein–Poisson bracket and get the following lemma.

Lemma A.9 (*Eilks [11]*). *The Rothstein–Poisson bracket on $\Gamma^\infty\left(\bigwedge^\bullet \tau^\#(L \oplus L^*)\right)$ is locally given by*

$$\begin{aligned} \{\phi, \psi\}_{\mathcal{R}} = {} & \nabla_{\frac{\partial}{\partial q^i}} \phi \wedge \frac{\partial}{\partial p_i} \psi - \frac{\partial}{\partial p_i} \phi \wedge \nabla_{\frac{\partial}{\partial q^i}} \psi + \tau^\# \left(R^\alpha_{\beta ij} a_\alpha \wedge a^\beta\right) \wedge \frac{\partial}{\partial p_i} \phi \wedge \frac{\partial}{\partial p_j} \psi \\ & + j(\tau^\# a_\alpha)\phi \wedge i(\tau^\# a^\alpha)\psi + j(\tau^\# a^\alpha)\phi \wedge i(\tau^\# a_\alpha)\psi, \end{aligned} \tag{A.23}$$

where $\psi, \phi \in \Gamma^\infty\left(\bigwedge^\bullet \tau^\#(L \oplus L^*)\right)$ and $\tau^\# a_\alpha$ and $\tau^\# a^\alpha$ are pullback basis sections.

From this formula we easily get the following lemma.

Lemma A.10. *For sections $s \in \Gamma^\infty(L)$, $\sigma \in \Gamma^\infty(L^*)$, $P \in \Gamma^\infty(\bigwedge^r L)$ and $\eta \in \Gamma^\infty(\bigwedge^s L^*)$ we have the equations*

$$\{\tau^\# s, \tau^\# \eta\}_{\mathcal{R}} = \tau^\#(i_s \eta) \tag{A.24}$$

$$\{\tau^\# \sigma, \tau^\# P\}_{\mathcal{R}} = \tau^\#(i_\sigma P) \tag{A.25}$$

and

$$\{\tau^{\#}s, \tau^{\#}P\}_{\mathcal{R}} = 0 = \{\tau^{\#}\sigma, \tau^{\#}\eta\}_{\mathcal{R}}. \tag{A.26}$$

In particular, we get

$$\{\tau^{\#}e_1, \tau^{\#}e_2\}_{\mathcal{R}} = \tau^{*}\langle e_1, e_2\rangle \tag{A.27}$$

for all $e_1, e_2 \in \Gamma^{\infty}(L \oplus L^*)$.

In this situation, the super-Darboux coordinates are given as follows:

Proposition A.11. *If we set*

$$r_i = p_i - \tau^{\#}\left(\Gamma_{i\alpha}^{\beta}\, a^{\alpha} \wedge a_{\beta}\right), \tag{A.28}$$

the only non-trivial Rothstein–Poisson brackets between the q^i, r_j, $\tau^{\#}a^{\alpha}$ *and* $\tau^{\#}a_{\beta}$ *are*

$$\{q^i, r_j\}_{\mathcal{R}} = \delta^i_j \quad \textit{and} \quad \{\tau^{\#}a^{\alpha}, \tau^{\#}a_{\beta}\}_{\mathcal{R}} = \delta^{\alpha}_{\beta}. \tag{A.29}$$

The grading with respect to the total degree can be refined in the following sense:

Definition A.12. Let $\deg_L$ and $\deg_{L^*}$ be defined by the local formula

$$\deg_L = p_i \frac{\partial}{\partial p_i} + \tau^{\#}a_{\alpha} \wedge i(\tau^{\#}a^{\alpha}) \quad \text{and} \quad \deg_{L^*} = p_i \frac{\partial}{\partial p_i} + \tau^{\#}a^{\alpha} \wedge i(\tau^{\#}a_{\alpha}). \tag{A.30}$$

For an element $\psi \in \mathcal{P}$ we say ψ has bidegree (r, s), if $\deg_L \psi = r\psi$ and $\deg_{L^*} \psi = s\psi$. The set of all such elements will be denoted by $\mathcal{P}^{(r,s)}$.

Of course we have $\deg = \deg_L + \deg_{L^*}$, and therefore we call deg the total degree. Moreover we have $\mathcal{P}^{(0,0)} = \tau^*(C^{\infty}(M))$, $\mathcal{P}^{(r,0)} = \tau^{\#}(\bigwedge^r L)$ and $\mathcal{P}^{(0,s)} = \tau^{\#}(\bigwedge^s L^*)$.

Lemma A.13. *The Rothstein–Poisson bracket restricted to the polynomial sections* $\mathcal{P}$ *is of bidegree* $(-1, -1)$*, i.e. for all* $\phi \in \mathcal{P}^{(r,s)}$, $\psi \in \mathcal{P}^{(t,u)}$ *we have* $\{\phi, \psi\}_{\mathcal{R}} \in \mathcal{P}^{(r+t-1,s+u-1)}$.

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